THE MAPLE PACKAGE FOR CALCULATING POINCARÉ SERIES.

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Abstract. We offer a Maple package `Poincare_Series` for calculating the Poincaré series for the algebras of invariants/covariants of binary forms, for the algebras of joint invariants/covariants of several binary forms, for the kernel of Weitzenböck derivations,for the bivariate Poincaré series of algebra of covariants of binary d-form and for the multivariate Poincaré series of the algebras of joint invariants/covariants of several binary forms.

1. Introduction

The Poincaré series of a graded algebra $A = \oplus_i (A)_i$ is defined as formal power series $\mathcal{P}(A, z) := \sum_{i=0}^{\infty} \dim(A)_i z^i$. If an algebra is finitely generated then its Poincaré series is the power series expansions of certain rational functions.

The present package implements results of the following papers:

- Leonid Bedratyuk, The Poincaré series of the covariants of binary forms, Int. Journal of Algebra,2010
- Leonid Bedratyuk,The Poincaré series of the joint invariants and covariants of the two binary forms, Linear and Multilinear algebra, 2010
- Leonid Bedratyuk, Linear locally nilpotent derivations and the classical invariant theory, I: The Poincare series, Serdica Math. J.,2010
- Leonid Bedratyuk, Bivariate Poincaré series for the algebra of covariants of a binary form, preprint arXiv:1006.1974
- Leonid Bedratyuk, Multivariate Poincaré series for the algebras of joint invariants and covariants of several binary forms, in preparation.

2. Instalation.

The package can be downloaded from the web http://sites.google.com/site/bedratyuklp/.

(1) download the file `Poincare_Series.mpl` and save it into your Maple directory;
(2) download the Xin's file (see a link at the web page) `Ell2.mpl` and save it into your Maple directory;
(3) run Maple;
(4) `> read "Poincare_Series.mpl":  read "Ell2.mpl":`
(5) If necessary use `> Help();`

3. Formulas for the Poincaré series.

Below are the list of main formulas.

3.1. **Invariants and covariants of binary form.** Let $\mathcal{I}_d$, $\mathcal{C}_d$ be algebras of invariants and covariants of binary d-form graded under degree. We have

$$\mathcal{P}(\mathcal{I}_d, z) = \sum_{0\leq k<d/2} \varphi_{d-2\,k}\left(\frac{(-1)^k z^{k(k+1)}(1-z^2)}{(z^2,z^2)_k\,(z^2,z^2)_{d-k}}\right), \text{ (Springer's formula)}, \tag{1}$$

$$\mathcal{P}(\mathcal{C}_d, z) = \sum_{0\leq k<d/2} \varphi_{d-2\,k}\left(\frac{(-1)^k z^{k(k+1)}(1+z)}{(z^2,z^2)_k\,(z^2,z^2)_{d-k}}\right), \tag{2}$$

here $(a,q)_n = (1-a)(1-a\,q)\cdots(1-a\,q^{n-1})$ denotes the q-shifted factorial and the function $\varphi_n : \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ defined by

$$\varphi_n\left(\sum_{i=0}^{\infty} a_i z^i\right) = \sum_{i=0}^{\infty} a_{in} z^i.$$

3.2. **Joint invariants and covariants of binary form.** Let $\mathcal{I}_{\boldsymbol{d}}, \mathcal{C}_{\boldsymbol{d}}$, $\boldsymbol{d} = (d_1, d_2, \dots, d_n)$ be algebras of joint invariants and joint covariants of n binary forms of degrees d_1, d_2, $\dots, d_n$. Then

$$\mathcal{P}(\mathcal{I}_{\boldsymbol{d}}, z) = \sum_{i=0}^{d^*}\sum_{k=1}^{\beta_i} \frac{1}{(k-1)!}\frac{d^{k-1}\left(z^{k-1}\varphi_{d^*-k}((1-z^2)\,A_{i,k}(z))\right)}{dz^{k-1}}, \tag{3}$$

$$\mathcal{P}(\mathcal{C}_{\boldsymbol{d}}, z) = \sum_{i=0}^{d^*}\sum_{k=1}^{\beta_i} \frac{1}{(k-1)!}\frac{d^{k-1}\left(z^{k-1}\varphi_{d^*-k}((1+z)\,A_{i,k}(z))\right)}{dz^{k-1}}, \tag{4}$$

$$A_{i,k}(z) = \frac{(-1)^{\beta_i-k}}{(\beta_i-k)!\,(z^i)^{\beta_i-k}} \lim_{t\to z^{-i}} \frac{\partial^{\beta_i-k}}{\partial t^{\beta_i-k}}\left(f_d(tz^{d^*}, z)(1-tz^i)^{\beta_i}\right).$$

The integer numbers $\beta_i, i = 0, \dots, 2d^*$, $d^* := \max(d_1, d_2, \dots, d_n)$, are defined from the decomposition

$$f_{\boldsymbol{d}}(tz^{d^*}, z) = \left((1-t)^{\beta_0}(1-tz)^{\beta_1}(1-tz^2)^{\beta_2}\dots(1-tz^{2\,d^*})^{\beta_{2\,d^*}}\right)^{-1},$$

where

$$f_{\boldsymbol{d}}(t,z) = \left(\prod_{k=1}^{s}(tz^{-\,d_k}, z^2)_{d_k+1}\right)^{-1}.$$

3.3. **Joint invariants and covariants of linear and quadratic binary forms.** Let $d_1 = d_2 = \ldots = d_n = 1$, i.e. $\boldsymbol{d} = (1,1,\dots,1)$. Then

$$\mathcal{P}(\mathcal{I}_{\boldsymbol{d}}, z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(k-1)!}\frac{(n)_{n-k}}{(n-k)!}\frac{d^{k-1}}{dz^{k-1}}\left(\left(\frac{z}{1-z^2}\right)^{2n-k-1}\right) = \frac{\mathrm{N}_{n-2}(z^2)}{(1-z^2)^{2n-3}}, \tag{5}$$

where $\mathrm{N}_n(z)$ is the n-th Narayama polynomial

$$\mathrm{N}_n(z) = \sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} z^{k-1}.$$

and $(n)_m := n(n+1)\cdots(n+m-1)$, $(n)_0 := 1$ denotes the shifted factorial.

$$\mathcal{P}(\mathcal{C}_{\boldsymbol{d}}, z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(k-1)!}\frac{(n)_{n-k}}{(n-k)!}\frac{d^{k-1}}{dz^{k-1}}\left(\frac{(1+z)z^{2n-k-1}}{(1-z^2)^{2n-k}}\right), \tag{6}$$

Let $d_1 = d_2 = \ldots = d_n = 2,\ \boldsymbol{d} = (2,2,\ldots,2)$, then

$$\mathcal{P}(\mathcal{I}_{\boldsymbol{d}}, z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(n)_i (n)_{n-k-i} (1-z) z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}} \right), \tag{7}$$

$$\mathcal{P}(\mathcal{C}_{\boldsymbol{d}}, z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(n)_i (n)_{n-k-i} z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}} \right) = \tag{8}$$

$$= \frac{\displaystyle\sum_{i=0}^{n-1} \binom{n-1}{i}^2 (z^2)^i}{(1-z)^n (1-z^2)^{2n-1}}. \tag{9}$$

3.4. **Kernel of Weitzenbök derivation.** Denote by $\mathcal{D}_{\boldsymbol{d}}$ the Weitzenbök derivation (linear locally nilpotent derivation) with its matrix consisting of n Jordan blocks of size $d_1+1,\, d_2+1,\, \ldots,\, d_s+1$, respectively. Since $\ker \mathcal{D}_{\boldsymbol{d}} \cong \mathcal{C}_{\boldsymbol{d}}$ and the isomorphism preserve degrees then have that $\mathcal{P}(\ker \mathcal{D}_{\boldsymbol{d}}, z) = \mathcal{P}(\mathcal{C}_{\boldsymbol{d}}, z)$.

3.5. **Bivariate Poincare series for covariants of binary form.** The algebra $\mathcal{C}_d$ of covariants is a finitely generated bigraded algebra:

$$\mathcal{C}_d = (\mathcal{C}_d)_{0,0} + (\mathcal{C}_d)_{1,0} + \cdots + (\mathcal{C}_d)_{i,j} + \cdots,$$

where each subspace $(\mathcal{C}_d)_{i,j}$ of covariants of degree i and order j is finite-dimensional. We have

$$\mathcal{P}(\mathcal{C}_d, z, t) = \sum_{i=0}^{\infty} (\mathcal{C}_d)_{i,j} z^i t^j = \sum_{0 \le k < d/2} \psi_{d-2k} \left(\frac{(-1)^k t^{k(k+1)} (1-t^2)}{(t^2,t^2)_k\, (t^2,t^2)_{d-k}} \right) \frac{1}{1 - z t^{d-2k}}, \tag{10}$$

where $\psi_n : \mathbb{Z}[[t]] \to \mathbb{Z}[[t,z]], n \in \mathbb{Z}_+$ be a $\mathbb{C}$-linear function defined by

$$\psi_n(t^m) := \begin{cases} z^i t^j, & \text{if } m = n\,i - j, j < n, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mathcal{P}(\mathcal{C}_d, z, 0) = \mathcal{P}(\mathcal{I}_d, z)$ and $\mathcal{P}(\mathcal{C}_d, z, 1) = \mathcal{P}(\mathcal{C}_d, z)$.

3.6. **Multivariate Poincaré series.** The algebra $\mathcal{C}_{\boldsymbol{d}}$ is a finitely generated multigraded algebra under the multidegree-order:

$$\mathcal{C}_{\boldsymbol{d}} = (\mathcal{C}_{\boldsymbol{d}})_{\boldsymbol{m},0} + (\mathcal{C}_{\boldsymbol{d}})_{\boldsymbol{m},1} + \cdots + (\mathcal{C}_{\boldsymbol{d}})_{\boldsymbol{m},j} + \cdots,$$

where each subspace $(\mathcal{C}_{\boldsymbol{d}})_{\boldsymbol{d},j}$ of covariants of multidegree $\boldsymbol{m} := (m_1, m_2, \ldots, m_n)$ and order j is finite-dimensional. The formal power series

$$\mathcal{P}(\mathcal{C}_{\boldsymbol{d}}, z_1, z_2, \ldots, z_n, t) = \sum_{\boldsymbol{m},j=0}^{\infty} \dim((\mathcal{C}_{\boldsymbol{d}})_{\boldsymbol{m},j}) z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} t^j,$$

is called the multivariariate Poincaré series of the algebra of join covariants $\mathcal{C}_{\boldsymbol{d}}$.

The following formula holds:

$$\mathcal{P}(\mathcal{C}_{\boldsymbol{d}}, z_1, z_2, \ldots, z_n, t) = \underset{\ge 0}{\Omega} f_{\boldsymbol{d}} \left(z_1 (t\lambda)^{d_1}, z_2 (t\lambda)^{d_2}, \ldots, z_s (t\lambda)^{d_s}, \frac{1}{t\lambda} \right),$$

where

$$f_{\boldsymbol{d}}(z_1, z_2, \ldots, z_n, t) = \frac{1}{\displaystyle\prod_{k=1}^{n}\prod_{j=0}^{d_k}(1 - z_k t^{d_k - 2j})},$$

For the multivariariate Poincaré series of the algebra of join invariants $\mathcal{I}_{\boldsymbol{d}}$ we have

$$\mathcal{P}(\mathcal{I}_{\boldsymbol{d}}, z_1, z_2, \ldots, z_n) = \underset{=0}{\Omega} f_{\boldsymbol{d}}\left(z_1(t\lambda)^{d_1}, z_2(t\lambda)^{d_2}, \ldots, z_s(t\lambda)^{d_s}, \frac{1}{t\lambda}\right).$$

Here $\underset{\geq 0}{\Omega}$ and $\underset{=0}{\Omega}$ are the MacMahon's Omega operators.

4. Package Commands and Syntax

Command name: `INVARIANTS_SERIES`
Feature: Computes the Poincare series for the algebras of joint invariants for the binary forms of degrees $d_1, d_2, \ldots, d_n$.
Calling sequence: `INVARIANTS_SERIES`$([d_1, d_2, \ldots, d_n])$;
Parameters:

$[d_1, d_2, \ldots, d_n]$	-	a list of degrees of n binary forms.
n	-	an integer, $n \geq 1$.

Command name: `COVARIANTS_SERIES`
Feature: Computes the Poincare series for the algebras of joint covariants for the binary forms of degrees $d_1, d_2, \ldots, d_n$.
Calling sequence: `COVARIANTS_SERIES`$([d_1, d_2, \ldots, d_n])$;
Parameters:

$[d_1, d_2, \ldots, d_n]$	-	a list of degrees of n binary forms.
n	-	an integer, $n \geq 1$.

Command name: `KERNEL_SERIES`
Feature: Computes the Poincare series for the kernel of Weitzenböck derivation defined by n Jordan block of sizes $d_1+1,\ d_2+1,\ \ldots,\ d_n$.
Calling sequence: `KERNEL_SERIES`$([d_1, d_2, \ldots, d_n])$;
Parameters:

$[d_1, d_2, \ldots, d_n]$	-	a list of sizes of the n Jordan blocks.
n	-	an integer, $n \geq 1$.

Command name: `BIVARIATE_SERIES`
Feature: Computes the bivariate Poincare series for the algebra of covariants of binary form of degree d. Also, computes the bivariate Poincare series for the kernel of the basic Weitzenböck derivation.
Calling sequence: `BIVARIATE_SERIES`$([d])$;
Parameters:

d	-	the degree of binary form.

Command name: `MULTIVAR_COVARIANTS`
Feature: Computes the multivariate Poincaré series for the algebra of joint covariants for n binary forms of degrees $d_1, d_2, \ldots, d_n$.
Calling sequence: `MULTIVAR_COVARIANTS`$([d_1, d_2, \ldots, d_n])$;

Parameters:

$[d_1, d_2, \ldots, d_n]$	-	a list of degrees of n binary forms.
n	-	an integer, $n \geq 1$.

Command name: `MULTIVAR_INVARIANTS`
Feature: Computes the multivariate Poincaré series for the algebra of joint invariants for n binary forms of degrees $d_1, d_2, \ldots, d_n$.
Calling sequence: `MULTIVAR_INVARIANTS`$([d_1, d_2, \ldots, d_n])$;
Parameters:

$[d_1, d_2, \ldots, d_n]$	-	a list of degrees of n binary forms.
n	-	an integer, $n \geq 1$.

5. Examples

5.1. **Compute $\mathcal{P}(\mathcal{I}_6, z)$.** Use the command

```
> INVARIANTS_SERIES([6]);
```

$$\frac{z^8 + z^7 - z^5 - z^4 - z^3 + z + 1}{(z^6 + z^5 + z^4 - z^2 - z - 1)\,(z^6 + z^5 - z - 1)\,(-1 + z^2)\,(-1 + z)}$$

5.2. **Compute $\mathcal{P}(\mathcal{C}_6, z)$.** Use the command

```
> COVARIANTS_SERIES([6]);
```

$$\frac{z^{10} + z^8 + 3\,z^7 + 4\,z^6 + 4\,z^5 + 4\,z^4 + 3\,z^3 + z^2 + 1}{(z^6 + z^5 + z^4 - z^2 - z - 1)\,(z^6 + z^5 - z - 1)\,(-1 + z^2)\,(-1 + z)^3}$$

5.3. **Compute $\mathcal{P}(\mathcal{I}_{(1,2,3)}, z)$.** Use the command

```
> INVARIANTS_SERIES([1,2,3]);
```

$$\frac{z^{12} + z^9 + 2\,z^8 + 3\,z^7 + 3\,z^6 + 3\,z^5 + 2\,z^4 + z^3 + 1}{(-1 + z^4)^2\,(-1 + z^3)^2\,(-1 + z)\,(-1 + z^2)\,(z^4 + z^3 + z^2 + z + 1)}$$

5.4. **Compute $\mathcal{P}(\mathcal{C}_{(2,2,2)}, z)$.** Use the command

```
> COVARIANTS_SERIES([2,2,2]);
```

$$\frac{z^4 + 4\,z^2 + 1}{(-1 + z)^3\,(-1 + z^2)^5}$$

5.5. **Compute $\mathcal{P}(\ker \mathcal{D}_{(4)}, z)$.** Use the command

```
> KERNEL_SERIES([4]);
```

$$\frac{z^2 - z + 1}{(-1 + z^2)\,(-1 + z^3)\,(-1 + z)^2}$$

5.6. **Compute** $\mathcal{P}(\ker \mathcal{D}_{(1,1,1,2)}, z)$**.** Use the command

```
> KERNEL_SERIES([1,1,1,2]);
```

$$\frac{z^8 + 2\,z^7 + 7\,z^6 + 11\,z^5 + 11\,z^4 + 11\,z^3 + 7\,z^2 + 2\,z + 1}{(-1 + z^2)^3\,(-1 + z^3)^3\,(-1 + z)^2}$$

5.7. **Compute** $\mathcal{P}(\mathcal{C}_4, z, t)$**.** Use the command

```
> BIVARIATE_SERIES([4]);
```

$$\frac{t^4 z^2 - z t^2 + 1}{(-1 + z t^2)\,(-1 + z t^4)\,(-1 + z^2)\,(-1 + z^3)}$$

5.8. **Compute** $\mathcal{P}(\mathcal{C}_{(1,1,2)}, z_1, z_2, z_3, t)$**.** Use the command

```
> dd:=[1,1,2]:MULTIVAR_COVARIANTS(dd);
```

$$\frac{{z_2}^2{z_1}^2{z_3}^2t^2 + t z_3{z_2}^2z_1 - t z_3 z_2 - z_2 z_1 z_3 + z_2 z_1 t^2 z_3 + t z_3{z_1}^2 z_2 - z_1 t z_3 - 1}{(-1 + z_3 t^2)\,(-1 + {z_3}^2)\,(-1 + t z_2)\,(-1 + z_3{z_2}^2)\,(-1 + z_1 t)\,(-1 + {z_1}^2 z_3)\,(-1 + z_2 z_1)}$$

5.9. **Compute** $\mathcal{P}(\mathcal{I}_{(4,4)}, z_1, z_2, t)$**.** Use the command

```
> dd:=[4,4]:MULTIVAR_INVARIANTS(dd);
```

$$-\frac{{z_1}^4{z_2}^4 + {z_2}^2{z_1}^2 + 1}{(-1 + {z_2}^2)\,(-1 + {z_2}^3)\,(-1 + {z_1}^2 z_2)\,(-1 + z_1 z_2)\,(-1 + z_1{z_2}^2)\,(-1 + {z_1}^2)\,(-1 + {z_1}^3)}$$